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A GENERALIZATION OF DIRAC'S THEOREM

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Let G be an $(r+2)$-connected graph in which every vertex has degree at least d and which has at least $2d-r$ vertices. Then, for any path Q of length r and vertex y not on Q, there is a cycle of length at least $2d-r$ containing both Q and y.

1. Introduction

All graphs considered in this paper are simple. For basic graph-theoretic terminology, we refer the reader to [3]. The following is a well-known result of Dirac [4].

Theorem 1.1. (Dirac). *A 2-connected graph G, in which the degree of every vertex is at least d and which has at least $2d$ vertices, contains a cycle of length at least $2d$.* ▌

Since Dirac's original publication of the above theorem, there have been several papers which present new proofs and generalizations. Grötschel [7] shows that, under the conditions of Dirac's theorem, G contains a cycle of length at least $2d$ which passes through any given vertex. Moreover, Voss [14] has announced that, under these conditions, G has a cycle of length at least $2d$ which contains any two given vertices. In this paper we prove this result.

Voss and Zuluaga [15] note that Grötschel's main theorem implies the following result for 3-connected graphs.

Theorem 1.2. *Let G be a 3-connected graph, in which each vertex has degree at least d, and which has at least $2d-1$ vertices. Then for any pair of distinct vertices x and z in G, there is a path of length at least $2d-2$ whose endpoints are x and z.* ▌

We also improve this result by proving that, under the given hypothesis, there is a path of length at least $2d-2$ joining x and z which passes through a given vertex y.

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2. Paths in 2-connected graphs

Let G be a graph. We denote the set of vertices of G by $V(G)$, and the set of edges of G by $E(G)$. For any vertex x in G, the set of neighbours of x is $N(x)$, and the degree of x is $d(x)$. Let P be a path in G. For vertices x and z on P, we denote the section of P from x to z and including both x and z by $P[x, z]$. If P has not been previously defined, we use $P[x, z]$ to denote any path from x to z. Let x and z be vertices of G, and let Y be a subset of the vertices of G. An (x, Y, z)-*path* is an (x, z)-path P which includes every vertex of Y. When Y has exactly one vertex y, P will also be referred to as an (x, y, z)-path. An $(x, Y, z\colon d)$-*path* is an (x, Y, z)-path whose length is at least d. If Y is empty, we shall refer to P as an $(x, z\colon d)$-path. For a set of vertices S and a vertex y, the *restriction of S onto y*, written $S|y$, denotes the set S if $y \notin S$, and the set $\{y\}$ if $y \in S$.

Lemma 2.1 generalizes a result of Erdős and Gallai [6] (see also Lovász [12], exercise 10.19).

Lemma 2.1. *Let G be a 2-connected graph with at least three vertices, let u and v be distinct vertices of G, and let d be an integer. Suppose that every vertex of G, except possibly u and v, has degree at least d. If X is a set of at most two vertices for which there is a (u, X, v)-path, then there is a $(u, X, v\colon d)$-path in G.*

Proof. We proceed by induction on d. If $d \leqq 2$, the result is obvious. Thus, we may assume that $d \geqq 3$.

Suppose that $|X| \leqq 1$. If $|X|=0$, let x and y be any two adjacent vertices of $G-\{u, v\}$. If $|X|=1$, let $X=\{x\}$ and let y be any vertex of $G-\{u, v\}$ adjacent to x. By Menger's theorem [13], there are two disjoint paths from $\{u, v\}$ to $\{x, y\}$, and therefore, a $(u, \{x, y\}, v)$-path in G. Thus we may restrict our attention to the case in which $|X|=2$.

Let $X=\{x, y\}$ and let P be a (u, X, y)-path. Without loss of generality, we may assume that x precedes y on P. If every neighbour of X is on P, then, since $d(x) \geqq d$, the length of P is at least d. Thus, we may assume that x has some neighbour which is not on P. Similarly, we may assume that y has some neighbour which is not on P. Let H_x be a component of $G-V(P)$ which contains a neighbour of x, and let H_y be a component of $G-V(P)$ which contains a neighbour of y. (We allow the possibility that $H_x=H_y$.) For any vertex z of $G-V(P)$, let $n(z)$ denote the number of neighbours of z which lie on P. We shall begin by finding an $(x_1, x_2\colon d-n(x_2))$-path P_x in H_x such that x_1 is a neighbour of x, and $n(x_2)>0$.

Suppose, first, that H_x has no cutvertex. Let x_1 and x_2 be vertices of H_x chosen as follows:

(i) x_1 and x_2 are distinct if H_x has more than one vertex;
(ii) x_1 is a neighbour of x, and x_2 has a neighbour on P other than x; and
(iii) $n(x_2)$ is as large as possible, subject to (i) and (ii).

(We can find such a pair of vertices because G is 2-connected.) Every vertex of H_x, except possibly x_1, has degree at least $d-n(x_2)$. By induction, there is an $(x_1, x_2\colon d-n(x_2))$-path P_x in H_x.

Suppose, now, that H_x has a cutvertex. Let z be a neighbour of x in H_x, let B be an endblock of H_x not containing z as an internal vertex, and let b be the cutvertex of H_x in B. If some internal vertex of B has a neighbour on P other than x, choose x_2 to be such a vertex with $n(x_2)$ as large as possible, and let $x_1=z$. Thus, the degree of every vertex of B, with the possible exception of b, is at least $d-n(x_2)$. By induction, there is a $(b, x_2: d-n(x_2))$-path in B. This path, concatenated with any (x_1, b)-path in H_x, is an $(x_1, x_2: d-n(x_2))$-path in H_x.

Thus, we may assume that no internal vertex of B has a neighbour on P other than x. Note that each internal vertex of B has degree at least $d-1$ in B. Let x_1 be any internal vertex of B with $n(x_1)=1$, and let x_2 be any vertex of H_x which has some neighbour on P other than x. By induction, there is an $(x_1, b: d-1)$-path in B, and this path, together with any (b, x_2)-path in H_x, is an $(x_1, x_2: d-1)$-path in H_x.

Hence, in each case, we have shown that there is an $(x_1, x_2: d-n(x_2))$-path P_x in H_x such that x_1 is a neighbour of x, and x_2 has some neighbour on P other than x.

Suppose that x_2 has a neighbour on $P[u, y]$ other than x, and let x_3 be a neighbour of x_2 on $P[u, y]$ which is closest along P to x, without being equal to x. Then

$$P' = (P-E(P[x, x_3]))\cup\{xx_1, x_2x_3\}\cup P_x$$

is a $(u, \{x, y\}, v)$-path (see Figure 1). Furthermore,

$$|E(P')| \geqq (n(x_2)-2)+2+(d-n(x_2)) \geqq d.$$

Thus, P' is a $(u, \{x, y\}, v: d)$-path.

Therefore, we may assume that every neighbour of x_2 on P is in the set $\{x\}\cup V(P[y, v])-y$ (see Figure 2).

Suppose $H_y\neq H_x$. Then, as before, we can choose a $(y_1, y_2: d-n(y_2))$-path P_y in H_y, where y_1 is neighbour of y, and y_2 has some neighbour on P other than y. If y_2 has a neighbour on $P[x, v]-y$ then, as in the case for H_x, we can find a $(u, \{x, y\}, v: d)$-path. Thus we may assume that every neighbour of y_2 on P, other than y, lies on $P[u, x]-x$. Let x_3 be the neighbour of x_2 closest to y on $P[y, v]$, and let y_3 be the neighbour of y_2 closest to x on $P[u, x]$. Then

$$P' = P[u, y_3]y_3y_2P_yy_1yP[y, x]xx_1P_xx_2x_3P[x_3, v]$$

is a $(u, \{x, y\}, v: d)$-path (see Figure 3).

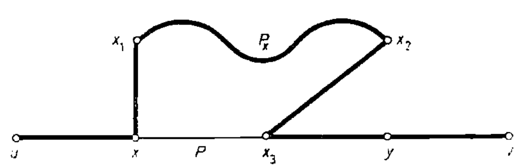

Fig. 1

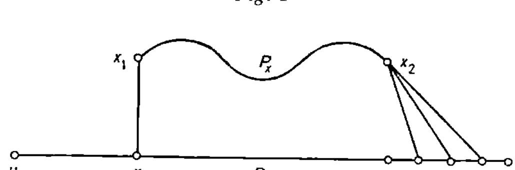

Fig. 2

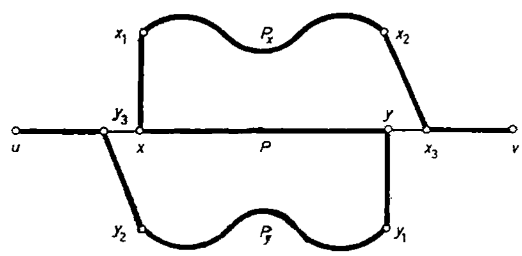

Fig. 3

In the remaining situation, $H_x = H_y$. Suppose H_x is 2-connected. Then, since x_2 is not a neighbour of y, we can choose any neighbour of y in H_y as y_1, and choose x_2 as y_2. By the induction hypothesis, there is a $(y_1, y_2\colon d-n(y_2))$-path P_y in H_y and, by the choice of x_2, no neighbour of y_2 lies on $P[u, y]-x$. Let y_3 be the neighbour of y_2 on P which is closest to y along P. Then

$$P' = (P-E(P[y, y_3]))\cup\{yy_1, y_2y_3\}\cup P_y$$

is a $(u, \{x, y\}, v\colon d)$-path.

If H_x is separable, let z, b and B be as defined earlier in this proof. Then, if B has an internal vertex adjacent to some vertex of P other than x, we can choose x_1 and x_2 as before. Let y_1 be a neighbour of y in H_x. If y_1 is not internal to B, then choose x_2 as y_2. Then there is a $(y_1, y_2\colon d-n(y_2))$-path P_y in H_x. Again, since y_2 has no neighbours on $P[u, y]-x$, there is a $(u, \{x, y\}, v\colon d)$-path in G.

If y_1 is internal to B, then, by the induction hypothesis, there is a $(b, y_1\colon d-n(x_2))$-path Q in B. Thus there is a $(x_1, y_1\colon d-n(x_2))$-path Q_x in H_x. Also, by the choice of x_2, $|E(P[y, v])|\geqq n(x_2)-1$. Hence

$$P' = P[u, x]\cup\{xx_1, y_1y\}\cup Q_x\cup P[y, v]$$

is a $(u, \{x, y\}, v\colon d)$-path.

Suppose that no internal vertex of B has a neighbour other than x on P. Let y_1 be any neighbour of y not in $B-b$, and let y_2 be any neighbour of x in $B-b$. As before, G has a $(u, \{x, y\}, v\colon d)$-path.

Thus, in all cases, we have shown that G has a $(u, \{x, y\}, v\colon d)$-path. ∎

In [9], we prove that the result of Lemma 2.1 holds if $|X|\leqq 3$.

3. Cycles in graphs of connectivity two

Using Lemma 2.1, it is now possible to prove a stronger version of Dirac's theorem for graphs with connectivity two. (In the next section, we shall prove a similar result for 3-connected graphs.) We call C an *$(x, y\colon d)$-cycle* if it contains x and y, and has length at least d.

Theorem 3.1. *Let G be a graph with connectivity two, let x and y be vertices of G, and let d be an integer, $d \geqq 2$. Suppose that each vertex of G has degree at least d. Then G has an $(x, y\colon 2d)$-cycle.*

Proof. Since G has connectivity two, there is a pair of vertices u and v in G such that $G-u-v$ is disconnected. We consider two cases depending on which components of $G-u-v$ contain x and y.

Case (i). Suppose that x and y are in different components of $G-u-v$, and let these two components be H_x and H_y, respectively. Let

$$G_x = G[V(H_x) \cup \{u, v\}] \cup \{uv\}$$

and

$$G_y = G[V(H_y) \cup \{u, v\}] \cup \{uv\}$$

Then G_x and G_y each satisfy the hypotheses of Lemma 2.1. Thus, there is a $(u, x, v\colon d)$-path P_x in G_x, and a $(u, y, v\colon d)$-path P_y in G_y. The cycle $P_x \cup P_y$ is an $(x, y\colon 2d)$-cycle in G.

Case (ii). We may assume that there is no pair of vertices u and v for which x and y fall into different components of $G-u-v$. We may assume that $G-x-y$ is connected, since otherwise we could choose the vertex cut $\{x, y\}$ and new vertices x' and y' of G such that x' and y' fall into different components of $G-x-y$, and then proceed as in Case (i). We may assume that $G-x$ is 2-connected, since otherwise we could choose a vertex cut $\{x, v\}$ and a new vertex x' of G such that x' and y fall into different components of $G-x-v$, and again, proceed as in Case (i). Similarly, we may assume that $G-y$ is 2-connected.

Let $\{u, v\}$ be any vertex cut of G, let H_1 be the component of $G-u-v$ containing x and y, and let H_2 be any other component. Let

$$G_1 = G[V(H_1) \cup \{u, v\}] \cup \{uv\}$$

and

$$G_2 = G[V(H_2) \cup \{u, v\}] \cup \{uv\}$$

Then G_2 satisfies the conditions of Lemma 2.1, and thus there is a $(u, v\colon d)$-path P_2 in G_2. We now want a $(u, \{x, y\}, v\colon d)$-path in G_1, and thus need only show that G_1 satisfies the conditions of Lemma 2.1; specifically, we must show that there is a $(u, \{x, y\}, v)$-path in G_1.

Suppose x and y are adjacent. Then, by Menger's theorem, there are disjoint paths Q_1 and Q_2 from $\{x, y\}$ to $\{u, v\}$. Thus, $Q_1 \cup Q_2 \cup \{x, y\}$ is a $(u, \{x, y\}, v)$-path in G_1.

We may therefore assume that x and y are not adjacent and that, in G, there is no vertex cut of size two separating x and y, and thus, by Menger's theorem, there is a set of internally-disjoint (x, y)-paths Q_1, Q_2 and Q_3. Since G_1 is 2-connected, there are two disjoint paths, $R_1[u, w_1]$ and $R_2[v, w_2]$ with $w_1, w_2 \in V(Q_1 \cup Q_2 \cup Q_3)$ (If $u \in V(Q_1 \cup Q_2 \cup Q_3)$, we set $R_1 = u$, and if $v \in V(Q_1 \cup Q_2 \cup Q_3)$, we set $R_2 = v$.) If w_1 and w_2 are both on the same path, say, Q_1 then

$$P_1 = R_1 \cup R_2 \cup Q_2 \cup (Q_1 - E(Q_1[w_1, w_2])$$

is a $(u, \{x, y\}, v)$-path in G_1. If w_1 and w_2 are on different paths, say, Q_1 and Q_2 respectively, then

$$P_1 = R_1 \cup R_2 \cup Q_3 \cup Q_1[x, w_1] \cup Q_2[w_2, y]$$

is a $(u, \{x, y\}, v)$-path in G_1. Thus, G_1 satisfies the conditions of Lemma 2.1. Hence, there is a $(u, \{x, y\}, v\colon d)$-path P_1' in G_1, and an $(x, y\colon 2d)$-cycle $P_1' \cup P_2$ in G. ∎

4. Paths and cycles in 3-connected graphs

In this section, we prove two closely related theorems: that a 3-connected graph G with minimum degree at least d contains

(i) an $(x, y\colon 2d)$-cycle C for any distinct vertices x and y in G, provided that $|V(G)| \geqq 2d$; and

(ii) an $(x, y, z\colon 2d-2)$-path P for any distinct vertices x, y and z in G, provided that $|V(G)| \geqq 2d-1$.

In the remainder of this paper, we call T a *trail* if T is a path or a cycle. This is a restriction of the usual definition of a trail. Let $S = s_1, s_2, \dots, s_k$ be a sequence of vertices, where $k \geqq 2$. We call T an $(s_1, s_2, \dots, s_k)$-*trail* if $s_1 \neq s_k$ and T is an (s_1, s_k)-path containing S in the specified order, or if $s_1 = s_k$ and T is a cycle containing S in the specified order. An $(s_1, s_2, \dots, s_k\colon d)$-*trail* is an $(s_1, s_2, \dots, s_k)$-trail of length at least d. The cycle C in (i) and the path P in (ii) are (x, y, z)-trails, with $x = z$ and $x \neq z$, respectively. The difference in the lengths that may be achieved occurs precisely because of this distinction, as we shall observe later.

We shall make use of the following lemma, which is similar to Lemma 2 of [2]. In the statement of Lemma 4.1, δ_x^z denotes a variation of the Kronecker delta: $\delta_x^z = 0$ if $x = z$, and $\delta_x^z = 1$ otherwise.

Lemma 4.1. *Let x and z be vertices of a graph G. Suppose that G has at least one (x, z)-trail, and let T be the longest such trail. Furthermore, suppose that there exist vertices u and v adjacent to T, but not on T, and a vertex q such that:*

(i) $|N(u) \cap N(v) \cap V(T)| = r$;

(ii) $|(N(u) \cup N(v)) \cap V(T)| = k$, *for some* $k \geqq 2$; *and*

(iii) there is a $(u, q, v\colon l)$-path P disjoint from T.

Then G contains an $(x, q, z\colon 2(k - \delta_x^z) + sl)$-trail, where

$$s = \begin{cases} 1 & \text{if } 0 \leqq r \leqq 1; \\ r-1 & \text{if } 2 \leqq r \leqq k. \end{cases}$$

Proof. Denote the neighbours of u and v on T by $v_1, v_2, \dots, v_k$, ordered along T. We shall construct $k-1$ (x, z)-trails $T_1, T_2, \dots, T_{k-1}$ corresponding to the $k-1$ pairs of vertices v_i, v_{i+1}, $i = 1, 2, \dots, k-1$.

There are two cases (see Figure 4).
If v_i and v_{i+1} are both adjacent to the same endvertex of P, say u, but not to the other, we define the corresponding trail T_i to be $T[x, v_i]v_i u v_{i+1} T[v_{i+1}, z]$. Otherwise, without loss of generality, v_i and v_{i+1} are adjacent to u and v, respectively. In this case we define the trail T_i to be $T[x, v_i]v_i u P v v_{i+1} T[v_{i+1}, z]$.

Since each T_i is an (x, z)-trail, and T is the longest such trail, the vertices v_i and v_{i+1} must be separated by at least two edges of T in the first case, and at least $2+l$ edges of T in the second case. Furthermore, at least s of the T_i contain P. Suppose v_i is adjacent to u, and v_{i+1} is adjacent to v, as in the second case. Then, if $x \neq z$, T_i has length at least $2(k-1)+sl$.

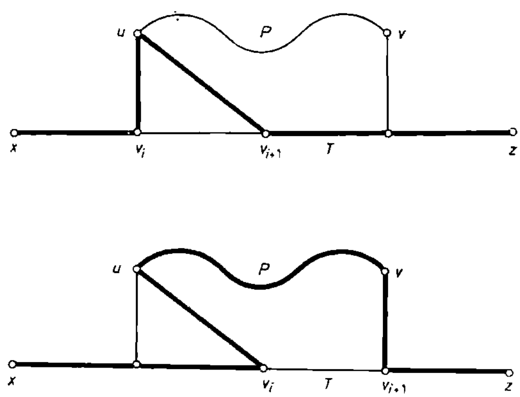

Fig. 4

If $x=z$, then either there is an edge from x to one of u or v, and thus we may count z as being v_{k+1}; or there is no edge from x to u or v, and thus there are two more edges in T_i. Hence, in either of these situations, $|E(T_i)| \geqq 2k+sl$. ∎

We note that the difference between the two cases $x=z$ and $x \neq z$ arises because the vertices $v_1, v_2, \dots, v_k$ must partition T into k subpaths if T is a cycle, but may only partition T into $k-1$ subpaths if T is a path.

In the next lemma, we require that stronger conditions be satisfied.

Lemma 4.2. *Let x and z be vertices of a graph G. Suppose that G has at least one (x, z)-trail, and let T be a longest such trail. Suppose that there exist vertices u_1, u_2 and u_3 adjacent to T, but not on T, and a vertex q such that*

(i) $|(N(u_1) \cup N(u_2) \cup N(u_3)) \cap V(T)| = k$, for some $k \geqq 3$;

(ii) if u_m and u_n both have exactly one neighbour on T, then these two neighbours are distinct, for $m \neq n$; and

(iii) there is a $(u_m, q, u_n$: $l)$-path P_{mn} disjoint from T, for $m \neq n$.

Then G has an $(x, q, z$: $2k+2l-2\delta_x^z)$-trail.

Proof. As in the proof of Lemma 4.1, let $v_1, v_2, \dots, v_k$ denote the neighbours of u_1, u_2 and u_3, and construct (x, z)-trails $T_1, T_2, \dots, T_{k-1}$, each of which uses some P_{mn}, if possible. At least two of these trails, say T_i and T_j, use some P_{mn} (not necessarily the same one). Thus, v_i and v_{i+1} are separated by at least $2+l$ edges on T, and v_j and v_{j+1} are separated by at least $2+l$ edges on T. Hence, $|T_i| \geqq \geqq 2(2+l)+2(k-2-\delta_x^z) = 2k+2l-2\delta_x^z$. ∎

Using the previous two lemmas, we now prove the main theorem of this section.

Theorem 4.3. *Let G be a 3-connected graph, let x, y and z be vertices of G, and let d be an integer,* $d \geqq 3$. *Suppose that G has at least* $2d-\delta_x^z$ *vertices, and that every vertex of G (except, possibly, x and z) has degree at least d. Then G has an* $(x, y, z\colon 2d-2\delta_x^z)$*-trail.*

Proof. For convenience, we set $m=2d-2\delta_x^z$. Let T be a longest (x, z)-trail in G. If T contains every vertex of G, then T is the required (x, y, z)-trail. Otherwise, let $y' \in (V(G)-V(T))|y$. We shall show that G has an $(x, y', z\colon m)$-trail T'. Thus if $y=y'$, T' is an $(x, y, z\colon m)$-trail. If $y \neq y'$, then $y \in V(T)$, but T is at least as long as T' and thus T is an $(x, y, z\colon m)$-trail.

Let H be the component of $G-V(T)$ containing y'. We consider four cases:
(i) $V(H)=\{y'\}$;
(ii) $V(H)=\{y', w\}$, for some vertex w;
(iii) H is separable;
(iv) H is 2-connected.

Case (i). All of the conditions for Lemma 4.1 are satisfied by setting $u=v=q=y'$, $l=0$, and $r=d(y') \geqq d$. Thus G has an $(x, y, z\colon m)$-trail.

Case (ii). All of the conditions for Lemma 4.1 are satisfied by setting $u=q=y'$, $v=w$, $l=1$, $k \geqq d(y')-1 \geqq d-1$ (note that $k \geqq 3$, since G is 3-connected) and $r \geqq 2(d-1)-k$. By Lemma 4.1, there is an (x, y', z)-trail T' such that

$$\begin{aligned} |E(T')| &\geqq 2(k-\delta_x^z)+s \\ &\geqq 2(k-\delta_x^z)+r-1 \\ &\geqq k-2\delta_x^z+2(d-1)-1 \\ &\geqq 2(d-\delta_x^z)+(k-3) \\ &\geqq m. \end{aligned}$$

Case (iii). Choose endblocks B_1 and B_2 of H, with cutvertices b_1 and b_2, respectively, such that there is either a (b_1, y', b_2)-path Q or a (y', b_1, b_2)-path Q' in H. (The cutvertices b_1 and b_2 are not necessarily distinct.) Let u be an internal vertex of B_1 with the largest number of neighbours on T, let d_1 denote this number, and let u' be any neighbour of u on T. Let v be an internal vertex of B_2 which has some neighbour on T other than u' and, subject to this condition, the largest number of neighbours, d_2, on T.

Let $y'' \in (V(B_1)-b_1)|y'$. By Lemma 2.1, there is a $(u, y'', b_1\colon d-d_1)$-path Q_1 in B_1 and a $(b_2, v\colon d-d_2)$-path in B_2. Let $P=Q_1QQ_2$ if $y' \in B_1$, or $P=Q_1Q'[b_1, b_2]Q_2$ if $y' \notin B_1$. Then the conditions of Lemma 4.1 are satisfied with this path P, $l \geqq 2d-d_1-d_2$ and $k \geqq \max\{d_1, d_2\}$. Hence, G has an (x, y', z)-trail T', with $|E(T')| \geqq 2(k-\delta_x^z)+l \geqq m$.

Case (iv). Since G is 3-connected there are three disjoint paths from $V(H)$ to $V(T)$. Thus there are vertices u_1, u_2 and u_3 satisfying conditions (i) and (ii) of Lemma 4.2. Choose such vertices u_1, u_2 and u_3 with d_1, d_2 and d_3 neighbours on T, respectively, such that $d_1 \geqq d_2 \geqq d_3$ and d_1 is maximum. Suppose, first, that $u_4 \in V(H) - \{u_1, u_2, u_3\}$ has d_4 neighbours on T, where $d_4 > d_1$. If $d_4 \geqq 3$, then u_4, u_2 and u_3 satisfy condition (i) and (ii) of Lemma 4.2, contradicting the choice of u_1, u_2 and u_3. Thus $d_4=2$, $d_1=d_2=d_3=1$, and $N(u_4)\cap V(T)=(N(u_2)\cup N(u_3))\cap V(T)$. Hence, u_4, u_1 and u_2 satisfy conditions (i) and (ii) of Lemma 4.2, again contradicting the choice of u_1, u_2 and u_3. We may therefore assume that u_1, u_2 and u_3 are vertices of H such that no vertex of $V(H)-\{u_1, u_2, u_3\}$ has more neighbours on T than u_1. Thus every vertex of H has degree at least $d-d_1$ in H. By Lemma 2.1, there is a $(u_i, y', u_j\colon d-d_1)$-path, $1 \leqq i < j \leqq 3$. Thus, by Lemma 4.2, there is a $(x, y', z\colon m)$-trail T' in G, completing the proof of the theorem. ▮

We separate the results of Theorem 4.3 into the following two corollaries.

Corollary 4.4. *Let G be a 2-connected graph with minimum degree d and at least $2d$ vertices, for some positive integer d. Then, for any two vertices x and y in G, there is a cycle of length at least $2d$ containing both x and y.*

Proof. Immediate from Theorem 3.1 and Theorem 4.3. ▮

Corollary 4.5. *Let G be a 3-connected graph with minimum degree d and at least $2d-1$ vertices, for some positive integer d. Then, for any three vertices x, y and z in G, there is an (x, z)-path of length at least $2d-2$ which contains y.*

Proof. Immediate from Theorem 4.3. ▮

Theorem 8 of [7] assumes conditions on the degree sequence of a graph. We state this theorem as it applies to $(r+2)$-connected graphs.

Theorem 4.6 (Grötschel). *Let G be an $(r+2)$-connected graph, $r \geqq 0$, with minimum degree d and at least $2d-r$ vertices. Then, for any path Q of length r, there is a cycle of length at least $2d-r$ which contains Q.* ▮

We can deduce a stronger result.

Corollary 4.7. *Let G be an $(r+2)$-connected graph, $r \geqq 0$, with minimum degree d and at least $2d-r$ vertices. Then, for any path Q of length r and vertex y not on Q, there is a cycle of length at least $2d-r$ which contains Q and y.*

Proof. If $r=0$, this corollary is equivalent to Corollary 4.4. If $r \geqq 1$, let S be the set of internal vertices of Q, and let x and z be the endpoints of Q. Then $G-S$ satisfies the hypotheses of Theorem 4.3 (with minimum degree $d-r+1$ and $x \neq z$). Thus, there is an $(x, y, z\colon 2(d-r+1)-2)$-path P in $G-S$. The cycle $P \cup Q$ has length at least $2d-r$. ▮

Enomoto [5] has recently proven an Ore-type generalization of Theorem 4.6.

5. Cycle space of 3-connected graphs

A graph G is *k-generated* if its cycle space is generated by cycles of length at least k. Theorem 4.6, with $r=1$, would be an immediate consequence of the following conjecture by Bondy [1].

Conjecture 5.1. *Let G be a 3-connected graph and let d be a positive integer. Suppose that each vertex of G has degree at least d, and that G has at least $2d$ vertices. Then, for every cycle C of G, there is a set of cycles $\{C_i\}_{i=1}^{m}$, where m is an odd integer and C_i has length at least $2d-1$ for each i, $i=1, 2, \dots, m$, such that*

$$C = \mathop{\Delta}_{i=1}^{m} C_i.$$

Conjecture 5.1 would imply that any graph satisfying these hypotheses is $(2d-1)$-generated. Let G be a graph on at least $2d$ vertices and with minimum degree at least d. Choose any two vertices x and y in G. Then $G\cup\{xy\}$ also satisfies the hypotheses of Conjecture 5.1. Hence, there would be a cycle C of length at least $2d-1$ using xy. But then $C-xy$ is an $(x, y\colon 2d-2)$-path in G.

Voss and Zuluaga [15] prove that every 2-connected graph with minimum degree d and at least $2d$ vertices has an even cycle of length at least $2d$ and, if G is not bipartite, an odd cycle of length at least $2d-1$. This result, restricted to 3-connected graphs, would be an immediate consequence of Conjecture 5.1.

Hartman [8] has proved the following theorem which is a variant of Conjecture 5.1 for 2-connected graphs. (We give a different proof in [11].)

Theorem 5.2 (Hartman). *Let d be a positive integer and let G be a 2-connected graph, $G\not\cong K_{d+1}$ if d is odd. Suppose that each vertex of G has degree at least d. Then G is $(d+1)$-generated.* ▮

In [10], we use methods similar to the methods of this paper to yield a partial proof of Conjecture 5.1. The variant we prove is:

Theorem 5.3. *Let G be a 3-connected graph and let d be a positive integer. Suppose that each vertex of G has degree at least d, and that either G has at least $4d-5$ vertices or G is non-hamiltonian. Then, G is $(2d-1)$-generated.* ▮

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